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Charged particle orbits in helical magnetic fields

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Abstract. It is shown that the radial motion of a charged particle moving in a helical coil may be described in terms of motion in a one-dimensional potential, the potential being related to a suitable average of the components of the magnetic field. An explicit form for this potential is given. Good agreement is found with numerical orbit calculations.

1. Introduction

In a recent paper Blewett and Chasman (1977) discussed the motion of a charged particle in a static magnetic field with helical symmetry. Such studies are of importance in the study of the production of synchrotron radiation using 'helical wiggler' devices and also in a study of the so-called 'free electron lasers'. Blewett and Chasman solved the particle orbit equations numerically and also developed an approximate analytic treatment which reproduced the main features of the numerical results. In their treatment the spatial variation of the magnetic field components was replaced by the lowest significant order Taylor series expansion. Thus their analysis is limited to those particles which remain close to the axis of the device for all their motion. These authors also made the reasonable assumption that the axial velocity was much larger than the radial component.

A little while ago the present author studied the motion of charged particles in a class of periodically varying magnetic fields (Rowlands 1975). For particles whose axial velocity is much larger than their radial component, a systematic expansion technique was developed which showed that, to lowest significant order, the transverse motion of the particle was identical to that of a particle constrained to move in a one-dimensional potential well. A general form for this potential was obtained in terms of a suitable average of the magnetic field components. Comparison of this theory with numerical results showed excellent agreement (Hooper 1975).

It is the purpose of this paper to describe the application of the above-mentioned expansion technique to the problem of a charged particle moving in a magnetic field with helical symmetry. Such fields are readily produced by current flowing in a solenoid whose turns are drawn apart to make a helix with finite pitch. Again one finds that the radial motion may be described by an effective one-dimensional motion in a potential well with the motion along the axis playing the role of time. The form of this potential is obtained as an average over one helical period of an explicit function of the associated vector potential. Knowledge of this potential is sufficient to understand the radial excursions of a charged particle in the real magnetic field. Unlike the treatment of Blewett and Chasman it is not necessary to limit the radial excursions of a particle to be near the axis.

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In the next section the exact equations of motion are manipulated into a form suitable for the expansion technique which is developed in § 3. A general form for the effective one-dimensional potential is given in § 3 and is applied to the case of a single helical coil in § 4.

The spatial variation of the components of a general static magnetic field with helical symmetry is discussed in an Appendix.

2. Exact equations of motion

The exact equation of motion of a relativistic particle in a static magnetic field may be written in the form

$$\mathrm{d}\beta/\mathrm{d}t = \boldsymbol{\beta} \wedge \boldsymbol{b},$$

where $\boldsymbol{\beta}$ is the ratio of the particle velocity to that of light and $\boldsymbol{b}$ a normalised magnetic field related to the actual magnetic field $\boldsymbol{B}$ by

$$\boldsymbol{b} = e\,\boldsymbol{B}/\gamma m_0 c.$$

Throughout this paper small letters denote the normalised magnetic fields corresponding to the actual magnetic fields denoted by capital letters. Here $\gamma^2 = 1/(1-\beta^2)$, m_0 is the particle rest mass, e is its charge and $\tau = ct$. In the Appendix it is shown that the various components of a helical field are related as follows. In a cylindrical coordinate system (r, θ, z), with vector potential $\boldsymbol{A} = (A_r, A_\theta, 0)$, the field components are given by

$$B_r = k\frac{\partial A_\theta}{\partial \eta}, \qquad B_\theta = -k\frac{\partial A_r}{\partial \eta}, \qquad B_z = B_0 - krB_\theta,$$

where $\eta = \theta - kz$ and k is the inverse wavelength of the helical coils. Here B_0 is the value of B_z on axis and is independent of η. Further,

$$\frac{\partial B_r}{\partial \eta} = \frac{\partial}{\partial r}(rB_\theta) \qquad \text{and} \qquad B_\theta = -k\left(\frac{\partial}{\partial r}(rA_\theta) - rB_0\right)\Big/(1+k^2r^2).$$

Because of the assumed symmetry of the magnetic field the equations of motion reveal the following constant of motion:

$$p = r\beta_\theta + \beta_z/k + ra_\theta.$$

Here a_θ is the normalised vector potential. Since the magnetic field components are functions of r, η $(=\theta - kz)$ only, it is convenient to change variables from r, θ, z to r, η, in which case the equations of motion take the form

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = k^2 s^2 (p-g)\frac{\partial g}{\partial r} + rs^4[k^2(p-g) + \beta_\eta]^2 + b_0 rs^2\beta_\eta,$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} = \frac{k^2}{r^2}(p-g)\frac{\partial g}{\partial \eta} - \frac{2\beta_r s^2}{r}[k^2(p-g)+\beta_\eta] - \beta_r b_0/r,$$

where $g = ra_\theta$, $\beta_\eta = \mathrm{d}\eta/\mathrm{d}\tau$ and $s^2 = 1/(1+k^2r^2)$. The total energy $\gamma m_0 c^2$ is of course a constant, and this implies constancy of β^2 which, in the above notation, takes the form

$$\beta^2 = \beta_r^2 + s^2[k^2(p-g)^2 + r^2\beta_\eta^2]. \tag{2.1}$$

Finally the above equations are combined to specify the path of the particle by giving its radial coordinate r as a function of η. The final equation is a nonlinear ordinary differential equation relating the first and second derivatives in the following manner:

$$\beta_\eta^2 \frac{d^2 r}{d\eta^2} = k^2 s^2 (p-g)\frac{\partial g}{\partial r} + rs^4[(p-g)k^2+\beta_\eta]^2 + b_0 r s^2 \beta_\eta$$
$$- \frac{dr}{d\eta}\left(\frac{k^2}{r^2}(p-g)\frac{\partial g}{\partial \eta} - \frac{b_0}{r}\beta_\eta \frac{dr}{d\eta} - \frac{2\beta_\eta}{r}\frac{dr}{d\eta} s^2[(p-g)k^2+\beta_\eta]\right). \tag{2.2}$$

In principle, equation (2.1) may be used to express β_η in terms of r, η and $dr/d\eta$, noting that $\beta_r = dr/d\tau = \beta_\eta\, dr/d\eta$. However it is convenient in the following to treat these equations separately.

3. Multiple scale perturbation theory

In 'helical wigglers' the transverse velocity components are small compared with the component in the direction of the axis of the device. This implies that the variation of r with respect to η has two distinct space scales. Firstly there is the fast variation associated with the axial motion, and hence with the spatial variation of the magnetic field on the scale of the helical variation of the field, and secondly the much slower variation associated with the radial motion of the particle. The perturbation theory developed below is based on the reasonable assumption that these space variations are of a different order of magnitude.

It is convenient to express the basic equations in dimensional form by introducing the reduced variables $\phi = \beta_\eta/k\beta$, $\delta = pk/\beta$, $g = p\epsilon h$ and $y = kr$. Here $\epsilon = 1/kR_L$, and $R_L = \beta m_0 \gamma c/eB_c$ is the Larmor radius of the particle defined with respect to the total velocity in a representative uniform magnetic field B_c. In these reduced variables equations (2.1) and (2.2) take the form

$$1 = s^2\delta^2(1-\epsilon h)^2 + \phi^2[y^2 s^2 + (dy/d\eta)^2], \tag{3.1}$$

and

$$\phi^2 \frac{d^2 y}{d\eta^2} = s^2\delta^2 \epsilon \frac{dh}{dy}(1-\epsilon h) + ys^4[\phi + \delta(1-\epsilon h)]^2 + \alpha\epsilon y \phi s^2$$
$$- \frac{dy}{d\eta}\left(\frac{\epsilon\delta^2}{y^2}(1-\epsilon h)\frac{\partial h}{\partial \eta} - \frac{\alpha\epsilon\phi}{y}\frac{dy}{d\eta} - \frac{2\phi}{y}s^2\frac{dy}{d\eta}[\delta(1-\epsilon h)+\phi]\right), \tag{3.2}$$

where now $s^2 = 1/(1+y^2)$ and $\alpha = b_0/b_c$. Further, in terms of these reduced variables,

$$\beta_\theta = \beta y s^2[\phi + \delta(1-\epsilon h)] \tag{3.3}$$

and

$$\beta_z = \beta s^2[\delta(1-\epsilon h) - y^2\phi]. \tag{3.4}$$

The basic expansion parameters are the ratios of the radial and azimuthal velocities to the axial velocity, that is β_r/β_z and β_θ/β_z. If we take the time dependence of β_θ and β_r to be related to the Larmor frequency, then both the above ratios are proportional to ϵ and we can thus treat ϵ as the expansion parameter. To this end we write

$$\delta = -1 + \epsilon\bar{\delta},$$

and expand ϕ and y as multiple scale functions of η, such that

$$y = y_0(\eta, \eta_1, \eta_2, \ldots) + \epsilon y_1(\eta, \eta_1, \ldots) + \epsilon^2 y_2 + \ldots$$

and

$$\phi = 1 - \epsilon\phi_1(\eta, \eta_1, \ldots) - \epsilon^2\phi_2(\eta, \eta_1, \ldots) + \ldots$$

where $\eta_1 = \epsilon\eta$, $\eta_2 = \epsilon^2\eta$, etc., as is usual in a multiple scale expansion (Nayfeh 1973).

Finally we need to expand h, which, from the form given in the Appendix, may be written in the form $h = -\hat{h}(y, \eta)/\delta$, so that

$$h(y, \eta) = \hat{h}(y_0, \eta) + \epsilon(y_1\partial h/\partial y_0 + \bar{\delta}\hat{h}) + \ldots.$$

Substitution of these expansions into equation (3.2) gives to lowest order

$$\partial^2 y_0/\partial\eta^2 = 0.$$

We take y_0 to be independent of η, since the solution proportional to η corresponds to an unbounded orbit. To next order one obtains

$$\partial^2 y_1/\partial\eta^2 = s_0^2\partial\hat{h}/\partial y_0 + \alpha y_0 s_0^2, \tag{3.5}$$

whilst the next order gives

$$\begin{aligned}\frac{\partial^2 y_2}{\partial\eta^2} = &-\left(\frac{\partial^2 y_0}{\partial\eta_1^2} + 2\frac{\partial^2 y_1}{\partial\eta_1\,\partial\eta} - 2\phi_1\frac{\partial^2 y_1}{\partial\eta^2}\right) + s_0^2\left(\frac{\partial\hat{h}}{\partial y_0}(-\bar{\delta} - \hat{h} - 2y_0y_1s_0^2) + \frac{\partial^2\hat{h}}{\partial y_0^2}y_1\right)\\ &+ s_0^4 y_0(\bar{\delta} + \hat{h} - \phi_1)^2 - \alpha y_1 s_0^4(1 - y_0^2) - \alpha s_0^2 y_0\phi_1\\ &-\left(\frac{\mathrm{d}y_1}{\mathrm{d}\eta} + \frac{\mathrm{d}y_0}{\mathrm{d}\eta_1}\right)\frac{1}{y_0^2}\frac{\partial\hat{h}}{\partial\eta}.\end{aligned} \tag{3.6}$$

From (3.1) one obtains the relation

$$\phi_1 = -(\hat{h} + \bar{\delta})/y_0^2 \tag{3.7}$$

whilst (3.3) and (3.4), to lowest significant order, take the form

$$\beta_\theta = \epsilon\beta(\hat{h} + \bar{\delta})/y_0, \tag{3.8}$$

and $\beta_z = -\beta + \mathrm{O}(\epsilon^2)$. Equation (3.7) may be used to eliminate ϕ_1 from (3.6), and this equation is then integrated over one helical period to give

$$\begin{aligned}\frac{\partial^2 y_0}{\partial\eta_1^2} = &-\frac{2}{y_0^2}\left\langle\hat{h}\frac{\partial^2 y_1}{\partial\eta^2}\right\rangle - s_0^2\left\langle\frac{\partial\hat{h}}{\partial y_0}, \bar{\delta} + \hat{h} + 2y_0y_1s_0^2\right\rangle\\ &+ s_0^2\left\langle y_1\frac{\partial^2\hat{h}}{\partial y_0^2}\right\rangle + \langle(\hat{h} + \bar{\delta})^2\rangle/y_0^3 + \alpha s_0^2\langle\hat{h} + \bar{\delta}\rangle/y_0 - \left\langle\frac{\partial y_1}{\partial\eta}\frac{\partial\hat{h}}{\partial\eta}\right\rangle/y_0^2\end{aligned} \tag{3.9}$$

where

$$\langle\ \rangle = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\eta.$$

We have, without loss of generality, taken $\langle y_1\rangle = 0$.

Using equation (3.5) the above equation may be written in the form

$$\partial^2 y_0/\partial\eta_1^2 = -\partial V/\partial y_0, \tag{3.10}$$

where

$$V(y_0) = \tfrac{1}{2}\langle(\partial y_1/\partial\eta)^2\rangle + (1/2y_0^2)\langle(\hat{h} + \bar{\delta})^2\rangle. \tag{3.11}$$

It is seen from (3.8) that the second term in the expression for V is simply equal to $\langle\beta_\theta^2\rangle/2\beta^2\epsilon^2$. This equation is the main result of the paper and of course is of the form of an equation for a particle moving in a one-dimensional potential $V(y_0)$, with η_1 playing the role of time. Thus the problem of the radial motion of a particle in a helical magnetic field can be reduced to a study of the motion in an effective one-dimensional potential. This potential is given in the form of an average over one helical period of a simple function of the θ component of the vector potential.

4. Particle orbits in a helical coil

The general form for the field due to such a coil is discussed in the Appendix, where it is shown that

$$\hat{h}(y,\eta)=\alpha\left[\left(2yy_a\sum_{m=1}^{\infty}\mathrm{K}_m'(my_a)\mathrm{I}_m'(my)\cos m\eta\right)-\tfrac{1}{2}y^2\right].$$

With this form for $\hat{h}$, equation (3.5) is readily solved for $\partial y_1/\partial\eta$ and this, together with equation (3.11), gives the general form for the potential V,

$$V(y)=\frac{\bar{\delta}^2}{2y^2}+s^4\alpha^2y_a^2\sum_{m=1}^{\infty}\left(\frac{\mathrm{K}_m'(my_a)}{m}\frac{\mathrm{d}}{\mathrm{d}y}[y\mathrm{I}_m'(my)]\right)^2$$
$$+\alpha^2y_a^2\sum_{m=1}^{\infty}[\mathrm{K}_m'(my_a)\mathrm{I}_m'(my)]^2+\tfrac{1}{8}\alpha^2y^2, \tag{4.1}$$

where without loss of generality we have identified B_0 and B_c so that $\alpha\equiv 1$.

In 'helical wigglers' two interweaving coils are wound so that the axial component of the magnetic field is zero on axis, that is $B_0\equiv 0$. For such coils the forms for $\hat{h}$ and V are as given above except that the term in each equation proportional to y^2 must be removed.

Blewett and Chasman (BC) in their analysis did not consider the full expression for the magnetic field, but only took the first term in the above series so that

$$\hat{h}=-2y(\bar{B}_0/B_c)\mathrm{I}_1'(y)\cos\eta$$

where $\bar{B}_0$ denotes the field strength defined by these authors and is simply related to B_0 used in the present paper by $\bar{B}_0=B_0y_a\mathrm{K}_1'(y_a)$. The corresponding effective potential is given by

$$V=\bar{\delta}^2/2y^2+2\omega^2[\![\{[y\mathrm{I}_1'(y)]'s^2\}^2+[\mathrm{I}_1'(y)]^2]\!]$$

where $2\omega^2=(\bar{B}_0/B_c)^2$. The problem is now to use this expression for $V(y)$ in equation (3.10) and solve for y_0 as a function of η_1. In particular, one may express the period of y_0 with respect to η_1 in the form

$$T=2\sqrt{2}\int_0^{y_m}\frac{\mathrm{d}y}{[V(y_m)-V(y)]^{1/2}}$$

where y_m is the maximum value of y. An approximate evaluation of this integral may be carried out in the following manner. For $\bar{\delta}\equiv 0$, the case considered in BC, and for $y\ll 1$

$$V\approx\tfrac{1}{2}\omega^2y^2(1+7y^2/32)+\mathrm{O}(y^6).$$

For more general y we replace V by its simplest Padé approximation and write

$$V = \tfrac{1}{2}\omega^2 y^2/(1-\alpha^2 y^2)$$

where α is a constant. With this form for V one finds

$$T = (4/\omega)(1-\alpha^2 y_{\mathrm{m}}^2)^{1/2} E(\alpha y_{\mathrm{m}})$$

where E is the complete elliptic integral of the second kind. Recalling that this is the period with respect to $\eta_1(=\epsilon(\theta - kz))$ and noting that equation (3.4) gives $z = \beta ct + \mathrm{O}(\epsilon^3)$, then it is readily found that the frequency of the oscillations with respect to time is given by

$$\bar{\omega} = \frac{c}{\sqrt{2}p} \bigg/ \frac{2E}{\pi}(1-\alpha^2 y_{\mathrm{m}}^2)^{1/2},$$

where p is the Larmor radius $(= m_0\gamma c/e\bar{B}_0)$ as defined in BC. Using the same parameters as BC, namely $p = 0{\cdot}348$ m, $y_{\mathrm{m}} = kr_{\mathrm{m}}$, $k = 196\ \mathrm{m}^{-1}$, $r_{\mathrm{m}} = 6{\cdot}2$ mm and $\alpha^2 = 7/32$, one finds $\bar{\omega} = 8{\cdot}1 \times 10^8\ \mathrm{s}^{-1}$. The computed value as quoted in BC is $7{\cdot}6 \times 10^8\ \mathrm{s}^{-1}$. Blewett and Chasman also obtain an analytic value for $\bar{\omega}$ by expanding the fields in powers of y, and they find

$$\bar{\omega} = (c/\sqrt{2}p)(1 + 3y_{\mathrm{m}}^2/32)$$

which gives, for the values quoted, $\bar{\omega} = 6{\cdot}9 \times 10^8\ \mathrm{s}^{-1}$. They actually state their value to be $7{\cdot}4 \times 10^8\ \mathrm{s}^{-1}$, but this is not consistent with their analytic formula. Their treatment of the fields and application to the problem considered is not really justifiable since they expand in powers of y, whereas the value of y is of order 1·2. The present treatment of course does not rely on such an expansion. Though the evaluation of the frequency of oscillations has been reduced to quadrature, it has not been found possible to evaluate this integral analytically. An approximate evaluation gives good agreement with the numerical results quoted in BC.

Summary

The problem of studying the motion of a charged particle in a helical magnetic field has been reduced to the study of the motion in a one-dimensional potential given explicitly by (3.11). For a particular field configuration the period of oscillation has been calculated and shown to be in good agreement with computer simulation results.

Appendix

We consider a current of magnitude J flowing in an infinitely long helical coil of pitch $2\pi/k$. In such a case all field components are functions of r and η $(= \theta - kz)$ only. In terms of the vector potential $\boldsymbol{A} = (A_r, A_\theta, 0)$ the field components are given by

$$B_r = k\ \partial A_\theta/\partial\eta, \qquad B_\theta = -k\ \partial A_r/\partial\eta, \qquad rB_z = \partial(rA_\theta)/\partial r - \partial A_r/\partial\eta.$$

Note we have not chosen a Lorentz gauge so $\nabla^2 \boldsymbol{A} \neq \boldsymbol{J}$. The condition that no current flows in the radial direction implies that $(\mathrm{curl}\ \boldsymbol{B})_r \equiv 0$, and this, with the condition that

$rB_\theta \to 0$ as $r \to 0$, gives rise to the relation

$$B_z + krB_\theta = B_0.$$

B_0 is a constant independent of η and equal to B_z on the axis. Because of the nature of the helical windings, we have that the θ and z components of the current are directly proportional, and this leads to the relation

$$\partial B_r/\partial \eta = \partial(rB_\theta)/\partial r.$$

Away from the current carrying coils one has

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(rA_r)\right) + \frac{(1+k^2r^2)}{r^2}\frac{\partial^2 A_r}{\partial \eta^2} = 0,$$

so that for $r < r_a$ (the radius of the coil)

$$rA_r = \sum_{m=0}^{\infty} C_m \mathrm{I}_m(mkr)\sin(m\eta).$$

The constants C_m may be obtained in the usual manner by matching the solutions for r less and greater than r_a, and in this way one obtains

$$B_\theta = \frac{kJy_a}{\pi y}\sum_{m=1}^{\infty} m\mathrm{K}'_m(my_a)\mathrm{I}_m(my)\cos m\eta.$$

Here $y = kr$, $y_a = kr_a$, K_m is the modified Bessel function of the second kind, I_m of the first kind and the prime denotes differentiation with respect to the argument. In the limit $r \to 0$ the above expression reduces to the one given by Smythe (1950 p 277). The value of B_z on axis, namely B_0, is also given by Smythe and takes the form

$$B_0 = Jk/2\pi.$$

The above results may be used to obtain all the components of the magnetic field, and in particular

$$rA_\theta(r, \eta) = (B_0/k^2)\left(\tfrac{1}{2}y^2 - 2yy_a\sum_{m=1}\mathrm{K}'_m(my_a)\mathrm{I}'_m(my)\cos m\eta\right).$$

Recalling that $\hat{h} = -rA_\theta k^2/B_c$ leads to the expression given for $\hat{h}$ in § 4.

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